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ON THE STABILITY OF THE STATIONARY MOTIONS OF SYSTEMS WITH FRICTION*

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The problem of the existence of the same stationary motions of systems with friction in an infinite time interval and of the corresponding systems with differential constraints is discussed together with their Lyapunov stability. Special attention is paid to the question of the relationship between the stability conditions of such motions in the case of systems with friction when the coefficient of friction is fairly high and in the case of non-holonomic systems. General conditions are illustrated by means of examples from the dynamics of a solid on a horizontal plane.

It is well-known /1-3/ that, under quite general assumptions, the motions of systems with friction transform in any finite interval of time into the motions of the corresponding non-holonomic systems as the coefficient of friction increases to infinity. The problem of the existence and stability of the stationary motions of the latter systems has been studied in detail (/4/, Chapter 4). However, the results /1-3/ do not enable one to use the theory /4/ to study the stability of the stationary motion of systems with friction since, as has already been noted, they only refer to a finite time interval while the Lyapunov stability characterizes the properties of the motion in an infinite time interval.

1. Let us recall the fundamental results of investigations of the stability of the stationary motions of non-holonomic systems (/4/, Chapter 4). Everywhere subsequently, the indices r, s and p take the values $1, \ldots, m$, the indices μ and ν take the values $m+1, \ldots, n$, the indices i and j take the values $1, \ldots, k$ and the indices $\alpha, \beta, \gamma, \delta$ take the values $k+1, \ldots, m$.

Let $q_1, \ldots, q_n$ be the generalized coordinates of the system, let the generalized velocities $q_1^{\cdot}, \ldots, q_n^{\cdot}$ be constrained by $n-m$ non-integrable relatinships of the form

$$q_\mu^{\cdot} = \sum_{r=1}^{m} b_{\mu r}(q) q_r^{\cdot} \tag{1.1}$$

and let $Q_1(q, q^{\cdot}), \ldots, Q_n(q, q^{\cdot})$ and $T(q, q^{\cdot})$ be the corresponding generalized forces and the kinetic energy of the system. Assuming, for simplicity, that the kinetic energy, generalized forces and the coefficients of the constraints are independent of the last $n-m$ coordinates q_μ while the coordinates of the force Q_μ corresponding to these coordinates are equal to zero, let us write down the equations of motion of the system being considered in the Chaplygin form

$$\frac{d}{dt}\frac{\partial T_*}{\partial q_r^{\cdot}} = \frac{\partial T_*}{\partial q_r} + Q_{r*} + \sum_{s,p=1}^{m} \Gamma_{rsp} q_s^{\cdot} q_p^{\cdot} \tag{1.2}$$

Here and henceforth, the subscript * denotes that the dependent velocities q_μ are eliminated with the help of relationship (1.1) from the corresonding expression, and the following notation is adopted:

$$T_* = \frac{1}{2}\sum_{r,s=1}^{m} \tau_{rs} q_r^{\cdot} q_s^{\cdot}, \quad \Gamma_{rsp} = \sum_{\mu=m+1}^{n} \nu_{\mu rs} \tau_{\mu p}$$

$$\tau_{\mu p} = \frac{\partial}{\partial q_p^{\cdot}}\left(\frac{\partial T}{\partial q_\mu^{\cdot}}\right)_*, \quad \nu_{\mu rs} = \frac{\partial b_{\mu r}}{\partial q_s} - \frac{\partial b_{\mu s}}{\partial q_r}$$

When the conditions

$$\frac{\partial Q_{r*}}{\partial q_\alpha} = \frac{\partial \tau_{rs}}{\partial q_\alpha} = \frac{\partial \Gamma_{rsp}}{\partial q_\alpha} = Q_{\alpha*} = \Gamma_{\alpha\beta\gamma} = 0 \tag{1.3}$$

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are satisfied, Eqs.(1.2) do not explicitly contain the coordinates q_α and allow of stationary solutions of the form

$$q_i = q_{i0}, \quad q_i^{\cdot} = 0, \quad q_\alpha^{\cdot} = q_{\alpha 0}^{\cdot} \tag{1.4}$$

if m constants $q_{i0}, q_{\alpha 0}^{\cdot}$ satisfy a system of $k < m$ equations

$$Q_{i*} + \sum_{\alpha, \beta = k+1}^{m} \left(\frac{1}{2} \frac{\partial \tau_{\alpha\beta}}{\partial q_i} + \Gamma_{i\alpha\beta} \right) q_\alpha^{\cdot} q_\beta^{\cdot} = 0 \tag{1.5}$$

The characteristic equation of a first-approximation system in the case of the equations of the perturbed motion of a non-holonomic system (1.2) in the neighbourhood of the stationary motion (1.4) has $m-k$ zero roots while its remaining roots satisfy the equation

$$\det \delta(\lambda) = 0 \tag{1.6}$$

$$\delta(\lambda) = \begin{Vmatrix} w_{ij}\lambda^2 + v_{ij}\lambda + u_{ij} & w_{i\beta}\lambda + v_{i\beta} \\ w_{\alpha j}\lambda + v_{\alpha j} & w_{\alpha\beta} \end{Vmatrix} \tag{1.7}$$

$$w_{rs} = (\tau_{rs})_0,$$

$$v_{rs} = \left[-\frac{\partial Q_{r*}}{\partial q_s^{\cdot}} + \sum_{\gamma=k+1}^{m} \left(\frac{\partial \tau_{r\gamma}}{\partial q_s} - \frac{\partial \tau_{s\gamma}}{\partial q_r} + \Gamma_{\gamma rs} - \Gamma_{rs\gamma} \right) q_\gamma^{\cdot} \right]_0$$

$$u_{ij} = -\left[\frac{\partial Q_{i*}}{\partial q_j} + \sum_{\gamma, \delta=k+1}^{m} \left(\frac{1}{2} \frac{\partial \tau_{\gamma\delta}}{\partial q_i \partial q_j} + \frac{\partial \Gamma_{i\gamma\delta}}{\partial q_j} \right) q_\gamma^{\cdot} q_\delta^{\cdot} \right]_0$$

Here and henceforth, the zero subscript indicates that the corresponding quantity is calculated on the unperturbed motion.

If the roots of (1.6) lie in the left half-plane, the unperturbed motion is stable and every perturbed motion sufficiently close to the unperturbed motion, tends asymptotically (as $t \to \infty$) to one of the stationary motions of the form of (1.4) belonging to the manifolds (1.6). If, however, at least one root of equation (1.6) lies in the right half-plane, the unperturbed motion is unstable.

2. Let us investigate whether stationary motions of the form (1.4), (1.5) exist subject to conditions (1.3) which satisfy relationships (1.1) in a mechanical system which is freed from the non-holonomic constraints (1.1) but subjected to the action of viscous friction forces derived from the Rayleigh function

$$F = -\frac{1}{2} \varkappa \sum_{\mu=m+1}^{n} \left(q_\mu^{\cdot} - \sum_{r=1}^{m} b_{\mu r} q_r^{\cdot} \right)^2$$

where $\varkappa > 0$ is the coefficient of friction (all quantities are assumed to be dimensionless) and, if such stationary motions exist, what are the conditions for their stability.

Let us introduce the quasivelocity

$$\pi_\mu^{\cdot} = -\sum_{s=1}^{m} b_{\mu s}(q) q_s^{\cdot} + q_\mu^{\cdot} \tag{2.1}$$

and pass from Lagrangian variables $q_1, \ldots, q_m, q_1^{\cdot}, \ldots, q_n^{\cdot}$ to the mixed variables $q_1, \ldots, q_m, q_1^{\cdot}, \ldots, q_m^{\cdot}, \pi_{m+1}, \ldots, \pi_n^{\cdot}$. In these new variables, the equations of motion of a system with friction take the form

$$\frac{d}{dt} \frac{\partial T^*}{\partial q_r^{\cdot}} = \frac{\partial T^*}{\partial q_r} + Q_r^* + \sum_{\mu=m+1}^{n} \frac{\partial T^*}{\partial \pi_\mu^{\cdot}} \sum_{s=1}^{m} \nu_{\mu rs} q_s^{\cdot} \tag{2.2}$$

$$\frac{d}{dt} \frac{\partial T^*}{\partial \pi_\mu^{\cdot}} = -\varkappa \pi_\mu^{\cdot}$$

Here, the superscript * denotes that the substitution (2.1) has been made in the corresponding expressions and the notation $\nu_{\mu rs}$ is identical to that introduced above.

Taking account of the obvious relationships

$$2T^* = 2T_* + 2 \sum_{\mu=m+1}^{n} \sum_{p=1}^{m} \tau_{\mu p} \pi_\mu^{\cdot} q_p^{\cdot} + \sum_{\mu, \nu=m+1}^{n} \tau_{\mu\nu} \pi_\mu^{\cdot} \pi_\nu^{\cdot}$$

$$Q_r^* = Q_{r*} + \Pi_r, \quad \Pi_r = \Pi_r(q_s, q_s^{\cdot}, \pi_\mu^{\cdot}), \quad \Pi_r(q_s, q_s^{\cdot}, 0) \equiv 0$$

system (2.2) can be represented in the form

$$\frac{d}{dt}\frac{\partial T_*}{\partial q_r^{\cdot}} = \frac{\partial T_*}{\partial q_r} + Q_{r*} + \sum_{s,\,p=1}^{m} \Gamma_{rsp} q_s^{\cdot} q_p^{\cdot} + \tag{2.3}$$

$$\Big[-\sum_{\nu=m+1}^{n} \pi_\nu^{\cdot\cdot}\tau_{\nu r} + \sum_{\nu=m+1}^{n}\sum_{s=1}^{m}\Big(\frac{\partial\tau_{\nu s}}{\partial q_r} - \frac{\partial \tau_{\nu r}}{\partial q_s}\Big) q_s^{\cdot}\pi_\nu^{\cdot} +$$

$$\Pi_r + \frac{1}{2}\sum_{\mu,\,\nu=m+1}^{n} \frac{\partial \tau_{\mu\nu}}{\partial q_r}\pi_\mu^{\cdot}\pi_\nu^{\cdot} + \sum_{\nu=m+1}^{n}\sum_{s=1}^{m}\Gamma_{rs\nu} q_s^{\cdot}\pi_\nu^{\cdot}\Big]$$

$$\sum_{s=1}^{m}\tau_{\mu s} q_s^{\cdot\cdot} + \sum_{\nu=m+1}^{n}\tau_{\mu\nu}\pi_\nu^{\cdot\cdot} + \sum_{s,\,p=1}^{m}\frac{\partial \tau_{\mu s}}{\partial q_p} q_s^{\cdot} q_p^{\cdot} +$$

$$\sum_{\nu=m+1}^{n}\sum_{s=1}^{m}\frac{\partial\tau_{\mu\nu}}{\partial q_s}\pi_\nu^{\cdot} q_s^{\cdot} = -\varkappa\pi_\mu^{\cdot}$$

$$\Big(\Gamma_{rs\nu} = \sum_{\mu=m+1} \nu_{\mu rs}\tau_{\mu\nu},\ \tau_{\mu\nu} = \frac{\partial^2 T}{\partial q_\mu^{\cdot}\partial q_\nu^{\cdot}}\Big)$$

It is obvious that the first m equations of system (2.3) only differ from Eqs. (1.2) in that there are terms contained in the square brackets which vanish when $\pi_\mu^{\cdot} = \pi_\mu^{\cdot\cdot} = 0$. However, conditions (1.3) are now insufficient for Eq. (2.3) not to contain explicitly the coordinates q_α and, also, stationary motions of the form of (1.4), (1.5) which satisfy (1.1)

$$q_i = q_{i0}, \quad q_i^{\cdot} = 0, \quad q_\alpha^{\cdot} = q_{\alpha 0}^{\cdot}, \quad \pi_\mu^{\cdot} = 0 \tag{2.4}$$

do not always exist but only when the supplementary relationships

$$\sum_{\alpha,\,\beta=k+1}^{m}\frac{\partial\tau_{\mu\alpha}}{\partial q_\beta} q_\alpha^{\cdot} q_\beta^{\cdot} = 0 \tag{2.5}$$

are satisfied.

Consequently, in the general case, the stationary motions of systems with viscous friction form manifolds of lower dimensionality than the stationary motions of the corresponding non-holonomic systems or do not exist at all (more precisely, they degenerate into equilibrium positions).

3. Let us assume that the coordinates q_α do not appear explicitly in the equations of motion of a system with friction, i.e. not only conditions (1.3) are satisfied but also the condition

$$\frac{\partial\tau_{\mu r}}{\partial q_\alpha} = \frac{\partial\tau_{\mu\nu}}{\partial q_\alpha} = \frac{\partial\Gamma_{rs\nu}}{\partial q_\alpha} = \frac{\partial\Pi_r}{\partial q_\alpha} = 0 \tag{3.1}$$

It is then obvious that a system with friction can execute stationary motions of the form of (1.4) of the corresponding non-holonomic system while the m constants q_{i0}, $q_{\alpha 0}^{\cdot}$ in (2.4) satisfy, as before, the system of $k < m$ Eqs. (1.5), since relationships (2.5) are automatically satisfied under conditions (3.1).

Let us now investigate the stability of these motions with respect to the variables

$$q_i,\ q_i^{\cdot},\ q_\alpha^{\cdot}.\ \pi_\mu^{\cdot}$$

The characteristic equation of the system for the first approximation in the case of the equations for the perturbed motion of a system with friction (2.3) in the neighbourhood of the stationary motion (2.4) has the form

$$\lambda^{m-k}\det\Delta(\lambda) = 0 \tag{3.2}$$

$$\Delta(\lambda) = \begin{Vmatrix} \delta_{ij}(\lambda) & \delta_{i\beta}(\lambda) & w_{i\nu}\lambda + v_{i\nu} \\ \delta_{\alpha j}(\lambda) & \delta_{\alpha\beta}(\lambda) & w_{\alpha\nu} \\ w_{\mu j}\lambda^2 + v_{\mu j}\lambda & w_\mu\lambda & w_{\mu\nu}\lambda + \varkappa e_{\mu\nu} \end{Vmatrix} \tag{3.3}$$

$$w_{\mu r} = w_{r\mu} = (\tau_{\mu r})_0, \quad w_{\mu\nu} = (\tau_{\mu\nu})_0, \quad e_{\mu\nu} = \begin{cases} 1 & (\mu = \nu) \\ 0 & (\mu \neq \nu)\end{cases}$$

$$w_{\mu j} = \Big(\sum_{\beta=k+1}^{m}\frac{\partial\tau_{\mu\beta}}{\partial q_j} q_\beta^{\cdot}\Big)_0, \quad v_{i\nu} = -\Big[\frac{\partial\Pi_i}{\partial\pi_\nu^{\cdot}} + \sum_{\beta=k+1}^{m}\Big(\frac{\partial\tau_{\nu\beta}}{\partial q_i} + \Gamma_{i\beta\nu}\Big) q_\beta^{\cdot}\Big]_0$$

($\delta_{rs}(\lambda)$ are the elements of the matrix $\delta(\lambda)$ (1.7)).

If at least one root of Eq.(3.2) lies in the right half-plane, the stationary motion (2.4) is unstable. If, however, all the roots of the equation

$$\det \Delta(\lambda) = 0 \tag{3.4}$$

lie in the left half-plane, it can be shown in a similar manner to that in /4/ that the special case of the critical case of several zero roots holds and that the Lyapunov-Malkin theorem is valid. When this is so, the stationary motion (2.4) is stable and every perturbed motion sufficiently close to the unperturbed motion tends asymptotically (as $t \to \infty$) to one of the stationary motions of th form of (2.4) belonging to the manifold (1.5).

Hence the question concerning the stability of the stationary motions of systems with friction reduces to an examination of the roots of Eq.(3.4). The question concerning the stability of the stationary motions of the corresponding non-holonomic systems reduces to an examination of the roots of (1.6) which is of lower degree than (3.4) and Eq.(3.4) therefore necessitates a calculation of the determinant (3.3) which is of higher order than (1.7).

4. Let us investigate whether it is, nevertheless, possible to reduce the examination of the roots of (3.4) to an investigation of the roots of Eq.(1.6) and, if this is possible, then under what conditions.

By taking out $\varkappa$ from the last $n-m$ rows of the determinant (3.3), we reduce Eq.(3.4) to the form

$$\varkappa^{n-m}[\det \delta(\lambda) + O(\varkappa^{-1})] = 0$$

whence it follows that the estimate

$$\lambda_h(\varkappa) = \lambda_h(\infty) + O(\varkappa^{-1}) \quad (h = 1, \ldots, 2k) \tag{4.1}$$

holds for the $2k$ roots of Eq.(3.4), where $\lambda_h(\infty)$ are the roots of Eq.(1.6).

Next, by making the substitution $\lambda = \varkappa l$ and taking out $\varkappa$ from the first k rows and columns and also from the last $n-m$ rows of the determinant $\det \Delta(\varkappa l)$, we reduce Eq.(3.4) to the form

$$\varkappa^{2k+n-m}\left[\det\begin{Vmatrix} w_{rs} & w_{r\nu} \\ w_{\mu s} & w_{\mu\nu}l + e_{\mu\nu}\end{Vmatrix} + O(\varkappa^{-1})\right] = 0$$

whence it follows that the estimate

$$\lambda_\mu(\varkappa) = -\varkappa l_\mu + O(1) \tag{4.2}$$

holds for the remaining $n-m$ roots of Eq.(3.4), where l_μ are the eigenvalues of the matrix

$$W = \left(w_{\mu\nu} - \sum_{r,s=1}^{m} w^{rs} w_{r\mu} w_{s\nu}\right)_{\mu,\nu=m+1}^{n} \quad ((w^{rs}) = (w_{rs})^{-1})$$

Since W is a matrix of absolutely-positive square form, all $l_\mu > 0$. Consequently, for sufficiently large $\varkappa > 0$, at least $n-m$ roots (4.2) of Eq.(3.4) always have negative real parts and the stability of the stationary motions of systems with friction depends on the real parts of just $2k$ roots (4.1) of this equation.

If $\varkappa \gg 1$ and all the roots of Eq.(1.6) have negative real parts (at least one root of Eq.(1.6) has a positive real part), then the remaining $2k$ roots (4.1) of Eq.(3.4) have negative real parts (at least one root of Eq.(3.4) has a positive real part). If, among the roots of Eq.(1.6), there are roots with a zero real part and no roots with a positive real part, the signs of the real parts of the roots (4.1) of Eq.(3.4) depend on small additions of the order of $\varkappa^{-1}$ when $\varkappa \gg 1$. The following assertion therefore holds.

Assertion. A stationary motion of a holonomic mechanical system with high viscous friction is stable and, at the same time, asymptotically stable with respect to some of the variables (exponentially unstable) if the stationary motion of the corresponding non-homonomic system is asymptotically stable with respect to some of the variables (exponentially unstable).

Corollary. Under the assumptions indicated above, the conditions for the stability of the stationary motions of holonomic systems with high viscous friction and for the corresponding non-holonomic systems are identical.

Remark. If a stationary motion of a non-holonomic system is non-asymptotically stable with respect to the variables characterizing the deviation of the perturbed motions from the manifold of stationary motions, then the stationary motion of the corresponding holonomic system with viscous friction can be both stable and, in particular, asymptotically stable with respect to some of the variables and, also, unstable. When this is so, the conditions for the stability of such a stationary motion of a system with friction may be quite different from the stability conditions for the stationary motion of the corresponding non-holonomic system.

5. As an example, let us consider the problem of the motion of a solid along a horizontal plane with viscous friction which, in the case of an infinite coefficient of friction, transforms into the problem of the motion of a solid along an absolutely rough surface (/4/, Chapter 5).

If the mass distribution of the body and the surface bounding it are arbitrary, then, in the case of an infinite value for the coefficient of friction, there exists a single parameter family of stationary motions of the body (permanent rotations) while, for any finite non-zero value of the coefficient of friction, there are only exists a zero-parameter family (positions of equilibrium). If, however, the mass distribution of the body is such that one of the principal central axes of inertia is orthogonal to its surface, the body can execute permanent rotations around an axis which is vertically arranged to the corresponding principal axis with an arbitrary angular velocity both on an absolutely rough surface as well as on a plane with friction (in this case conditions (2.5) are automatically satisfied).

In first case the characteristic equation of the perturbed motion has the form

$$\lambda f_4(\lambda) = 0; \quad f_4(\lambda) = \sum_{i=0}^{4} a_i \lambda^{4-i}$$

and, in the second case,

$$\lambda f_6(\lambda) = 0, \quad f_6(\lambda) = \sum_{j=0}^{6} b_j \lambda^{6-j}$$

(For explicit expressions for the coefficients a_i and b_j see /4, Chapter 5/).

When the relationships

$$\begin{aligned}
&[(J_2 - J_1)(r_2 - r_1)\sin 2\delta]\,\omega > 0 \\
&\{(J_1 + J_2 - J_3)(r_1 + r_2 - 2h) - mh\,[4h^2 - 3h(r_1 + r_2) + 2r_1r_2]\}\,\omega^2 - \\
&\quad mg(r_1 - h)(r_2 - h) > 0 \\
&\{(J_3 - J_1)(J_3 - J_2) + mh\,[(J_3 - B)(r_1 - h) + (J_3 - A)(r_2 - h)] + \\
&\quad m^2h^2(r_1 - h)(r_2 - h)\}\,\omega^4 + mg\,\{(J_3 - B)(r_1 - h) + (J_3 - A)(r_2 - \\
&\quad h) + 2mh(r_1 - h)(r_2 - h)\}\,\omega^2 + m^2g^2(r_1 - h)(r_2 - h) > 0 \\
&(A = J_1\cos^2\delta + J_2\sin^2\delta, \quad B = J_1\sin^2\delta + J_2\cos^2\delta)
\end{aligned} \tag{5.1}$$

are satisfied, all the roots of the equations $f_4(\lambda) = 0$ lie in the left half-plane. When there is a severe breakdown of at least one of the above-mentioned inequalities, the equation $f_4(\lambda) = 0$ has roots in the right half-plane. Consequently, if the coefficient of friction is sufficiently large, analogous assertions also hold for the roots of the equations $f_6(\lambda) = 0$. This has previously been shown /3/ by direct investigation of the roots of the equation $f_6(\lambda) = 0$.

Here J_1, J_2 and J_3 are the principal central moments of inertia of the body, m is the mass, g is the acceleration due to gravity, ω is the angular velocity, r_1 and r_2 are the principal radii of curvature of the surface of the body at the point where it touches the supporting plane, δ is the angle between the principal central axis of inertia corresponding to the moment J_1 and the principal direction corresponding to the radius r_1, and h is the height of the centre of mass.

Hence, the conditions for the stability of the permanent rotations of a heavy asymmetric body on an absolutely rough horizontal plane and on a plane with high visocous friction are identical. In particular, the stability of the rotation of a body depends, in both cases, on the direction of rotation (see (5.1)).

If, however, the body is symmetrical $(J_1 = J_2, r_1 = r_2)$, the function $f_4(\lambda)$ only contains even powers of λ and the equation $f_4(\lambda) = 0$ cannot have all roots with a negative real part. At the same time, $f_6(\lambda)$, as before, contains all powers of λ and the equation $f_6(\lambda) = 0$ can have all roots with a negative real part.

Hence, the stability conditions for the permanent rotations of a symmetrical solid on an absolutely rough surface and on a plane with friction can be substantially different and this difference does, in fact, occur. In the first case, the stability (instability) condition has the form /5/

$$(J_3 + mhr_1)^2\,\omega^2 - 4(J_1 + mh^2)\,mg\,(h - r_1) > 0 \quad (< 0) \tag{5.2}$$

and, in the second case, it has the form /6/

$$(J_3h - J_1r_1)\,\omega^2 - mg\,(h - r_1)\,h^2/r_1 > 0 \quad (< 0) \tag{5.3}$$

In particular, if the mass distribution of a top is such that $J_3h - J_1r_1 < 0$ (a Chinese top), the rapid rotation of the top with the lowest of the centre of mass $(h < r_1)$ is unstable on a plane with friction but stable on an absolutely rough plane.

We note that, if account is taken of the air resistance to the fall (but not to the rotation) of the top, the function $f_4(\lambda)$ will again contain all powers of λ and the equation $f_4(\lambda) = 0$ will have all roots with a negative real part (a root with a positive real part) provided that

$$[(J_3 + mhr_1) - (J_1 + mh^2)]\,\omega^2 - mg\,(h - r_1) > 0\ (< 0) \tag{5.4}$$

According to the results in paragraph 4, the inequality (5.4) also defines the stability condition for the rotation of a top on a plane with high viscous friction if the above-mentioned air resistance is taken into account in addition to the friction against the plane. We note that, unlike inequality (5.3), which is valid for any value of the coefficient of friction not equal to zero or infinity, inequality (5.4) is only valid when the value of this coefficient is fairly large. In the general case, the stability of the rotation of a top on a plane with friction allowing for air resistance is determined by a rather cumbersome inequality and depends on the ratio of the coefficients of sliding friction and the air resistance.

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IMPACTS IN A SYSTEM WITH CERTAIN UNILATERAL COUPLINGS*

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The characteristics of the dynamics of a system with ideal unilateral couplings resulting from the possibility of a simultaneous impact against two or more couplings are studied.

It is shown that a correct definition of an impact impulse during repeated impact is only possible in exceptional cases, that is, if the couplings are orthogonal or the impact is of an absolutely inelastic nature (in spite of the elasticity of each coupling individually). In the general case a percussive impulse does not possess the property of a continuous dependence on the initial conditions and the number of surfaces of discontinuity in phase space increases rapidly as the number of repetitions of the impact increases. In view of this, the problem of determining the post-impact motion in systems with a large number of unilateral couplings is of a stochastic nature.

The equations of motion are regularized in the case of orthogonal couplings and absolutely elastic collisions. Examples are considered which show the effect of the geometric and elastic properties of the couplings on the motion of certain mechanical systems.

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